A State-Space Approach for Adaptive Second-Order IIR Notch Filters with Constrained Poles and Zeros

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Abstract—This paper deals with a state-space approach for adaptive second-order IIR notch filters with constrained poles and zeros. A simplified adaptive iterative algorithm is derived from the gradient-descent method for minimizing the mean-squared output error of an adaptive notch filter. Stability and parameter estimation bias are analyzed for the adaptive iterative algorithm. A numerical example is presented to demonstrate the validity and effectiveness of the proposed adaptive state-space notch filter and parameter-estimation bias analysis.

I. INTRODUCTION

Adaptive notch digital filters are very useful to detect narrow-band signals such as sinusoids of unknown frequencies in a wide-band signal, and have many signal processing applications in digital communications, control, active noise control, radar, sonar, biomedical engineering, etc. [1]-[3]. Since adaptive IIR notch digital filters can be realized more efficiently than the FIR counterparts from the viewpoint of the number of coefficients and computational complexity, many methods for the adaptive IIR filtering have been proposed to date [4]-[18]. Among them, an adaptive IIR notch digital filter with constrained poles and zeros has been designed to eliminate multiple unknown sine waves embedded in a broad-band signal [4]. Gradient-based algorithms with IIR notch filtering have been employed to estimate sinusoids embedded in noise [7]. The performance (such as stability, estimation bias and mean square error) of a plain gradient algorithm has been analyzed for a gradient-based second-order adaptive IIR notch digital filter with constrained poles and zeros [15].

This paper is concerned with a state-space approach for adaptive second-order IIR notch filters with constrained poles and zeros. A simplified iterative algorithm that is derived from the gradient-descent method is applied to minimize the mean-squared output error of the filter. The stability and parameter estimation bias of the adaptive state-space notch digital filter are analyzed by utilizing a first-order linear dynamical system. Finally, a numerical example is included to illustrate the validity and effectiveness of the state-space notch digital filter and the parameter-estimation bias analyzed for the simplified adaptive iterative algorithm. It is also clarified that the present adaptive iterative algorithm and estimation-bias analysis are much simpler than those reported in [15].

II. A STATE-SPACE NOTCH DIGITAL FILTER AND ITS ADAPTIVE ITERATIVE ALGORITHM

A. Description of a notch digital filter

Consider a second-order IIR notch digital filter described by [4], [15]

$$H(z) = \frac{1 + az^{-1} + z^{-2}}{1 + \rho az^{-1} + \rho^2 z^{-2}}$$

where $a = -2 \cos \omega_c$ with notch frequency $\omega_c$ and $\rho$ is a pole radius which controls the 3-dB attenuation bandwidth of notch frequency and is chosen as $0 \leq \rho < 1$ for the filter to be stable.

The IIR notch digital filter in (1) can be realized by the following state-space model:

$$x(k+1) = Ax(k) + bu(k)$$
$$y(k) = cx(k) + du(k)$$

(2a)

where $x(k) = [x_1(k), x_2(k)]^T$ is a 2 × 1 state-variable vector, $u(k)$ is a scalar input, $y(k)$ is a scalar output, and $A, b, c$ and $d$ are real constant matrices given by

$$A = \begin{bmatrix} 0 & 1 \\ -\rho^2 & -\rho a \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$c = \begin{bmatrix} 1 - \rho^2 & (1 - \rho)a \end{bmatrix}, \quad d = 1.$$ (2b)

The transfer function of the state-space model in (2a) can be expressed as

$$H(z) = \frac{Y(z)}{U(z)} = c(zI_2 - A)^{-1}b + d$$

(3)

where $U(z)$ and $Y(z)$ denote the $z$-transforms of input $u(k)$ and output $y(k)$, respectively.

The transfer function from input $u(k)$ to state-variable vector $x(k)$ can be written as

$$\begin{bmatrix} F_1(z) \\ F_2(z) \end{bmatrix} = (zI_2 - A)^{-1}b$$
$$= \frac{1}{z^2 + \rho az + \rho^2} \begin{bmatrix} 1 \\ z \end{bmatrix}$$

(4a)

which leads to

$$X_2(z) = \frac{z^{-1}}{1 + \rho az^{-1} + \rho^2 z^{-2}} U(z)$$

(4b)

where $X_2(z)$ denotes the $z$-transform of $x_2(k)$ in state-variable vector $x(k)$. 

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From (1) and (3), it follows that
\[
\frac{\partial Y(z)}{\partial a} = \frac{\partial H(z)}{\partial a} U(z) = \frac{1 - \rho H(z)}{1 + \rho_a z^{-1} + \rho^2 z^{-2}} U(z) \tag{5a}
\]
\[
\frac{1 - \rho H(z)}{1 + \rho_a z^{-1} + \rho^2 z^{-2}} \tag{5b}
\]
where \(1 - \rho H(z)\) is a bandpass digital filter described by
\[
\frac{1 - \rho H(z)}{1 + \rho_a z^{-1} + \rho^2 z^{-2}}.
\]

B. Construction of adaptive state-space notch filters

Let the input signal to an adaptive notch digital filter be given by
\[
u(k) = \cos(\omega_o k + \theta) + v(k) \tag{6}
\]
where \(A\) and \(\theta\) denote the magnitude and phase of the sinusoid, respectively, \(\omega_o\) is the frequency of the sinusoid, and \(v(k)\) is the white noise of a Gaussian random process with zero mean and covariance \(\sigma_v^2\).

Referring to (2a) and (2b), an adaptive state-space notch digital filter can be described by
\[
x(k + 1) = A(k) x(k) + b u(k) \tag{7a}
\]
y(k) = c(k) x(k) + du(k)
where \(x(k) = [x_1(k), x_2(k)]^T\) is a state-variable vector and
\[
A(k) = \begin{bmatrix} 0 & 1 \\ -\rho^2 & -\rho a(k) \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{7b}
\]
c(k) = \begin{bmatrix} 1 - \rho^2 & (1 - \rho)a(k) \end{bmatrix}, \quad d = 1.

In order to tune the time-varying coefficient \(a(k)\), let us consider a simplified iterative algorithm (SIA) given by
\[
a(k + 1) = a(k) - \mu y(k) x_2(k) \tag{8}
\]
where \(x_2(k)\) can be viewed as an approximate value of signal \(\frac{\partial y(k)}{\partial a}\) in (5a), and found from the second entry of state-variable vector \(x(k) = [x_1(k), x_2(k)]^T\) in (7a).

In [19], [20] a variable step size (VSS) algorithm has been introduced for further improvement of the convergent performance in adaptive digital filters. The same idea is applied to the iterative algorithm in (8) as follows:
\[
a(k + 1) = a(k) - \mu y(k) x_2(k) \tag{9a}
\]
where
\[
\mu(k + 1) = \xi \mu(k) + \eta y(k)^2 \left( y(k - 1)^2 \right)^2 \quad 0 << \xi < 1 \quad \text{and} \quad 0 < \eta << 1. \tag{9b}
\]
Real numbers \(\xi\) and \(\eta\) are chosen close to unity and zero, respectively, so that \(\mu(k)\) remains large at the beginning of iteration and gradually decreases as iteration approaches the steady state. This will cause a smaller steady-state error than fixed \(\mu\) by choosing \(\xi, \eta\) and \(\mu(0)\) appropriately.

The block diagram of an adaptive state-space notch digital filter is illustrated in Fig. 1.

It should be noted that (5a) is written as
\[
\frac{\partial Y(z)}{\partial a} = \Delta H(z) z^{-1} [1 - \rho H(z)] U(z) \tag{10a}
\]
\[
= \frac{\Delta H(z) z^{-1} [U(z) - \rho Y(z)]}{1 + \rho_a z^{-1} + \rho^2 z^{-2}}. \tag{10b}
\]
The plain gradient iterative algorithm in [15] was given by
\[
a(k + 1) = a(k) - \mu y(k) [u(k - 1) - \rho y(k - 1)]. \tag{11}
\]
It is clear that the algorithm in (8) is simpler than that in (11).

III. Steady-State Analyses

This section investigates the estimation bias of parameter \(a(k)\) at steady state for the adaptive notch digital filter in noisy case. When the adaption algorithm sufficiently approaches its steady state, the filter parameter \(a(k)\) will be close to its true value \(a_0 = -2 \cos \omega_o\).

Applying the Taylor series expansion, the transfer function \(F_2(z)\) in (4a) in the vicinity of \(a_0\) can be approximated to
\[
F_2(e^{j\omega_o}) = \frac{e^{-j\omega_o}}{1 + \rho a(k)e^{-j\omega_o} + F^2 e^{-2j\omega_o}}
\]
\[
= \frac{1}{a_0(\rho - 1) + \rho \delta_a(k) + (\rho^2 - 1)e^{-j\omega_o}}
\]
\[
\simeq \frac{1}{a_0(\rho - 1) + (\rho^2 - 1)e^{-j\omega_o} + \rho^2 \delta_a(k) + [a_0(\rho - 1) + (\rho^2 - 1)e^{-j\omega_o}]^2 \delta_a^2(k)} \tag{12}
\]
where \(\delta_a(k) = a(k) - a_0\). The magnitude and phase of \(F_2(e^{j\omega_o})\) in (12) in the case of \(\delta_a(k) = 0\) are written as [15]
\[
B = \left| \frac{1}{a_0(\rho - 1) + (\rho^2 - 1)e^{-j\omega_o}} \right| \tag{13}
\]
\[
\phi = \begin{cases} 
\phi_0 & \text{for } \omega_o \leq \frac{\pi}{2} \\
\pi + \phi_0 & \text{for } \omega_o > \frac{\pi}{2}
\end{cases}
\]
where 
\[ \phi_0 = \tan^{-1} \frac{(1 + \rho)\sin\omega_o}{(1 - \rho)\cos\omega_o} \]

Hence, the signal \( x_2(k) \) can be described from (4) and (6) as
\[ x_2(k) = AB \cos(\omega_o k + \theta - \phi) - \rho AB^2 \delta_0(k) \cos(\omega_o k + \theta - 2\phi) + \rho^2 AB^3 \delta_0^2(k) \cos(\omega_o k + \theta - 3\phi) + v_2(k) \] (14)

where \( v_2(k) \) is a noise term in \( x_2(k) \) due to input noise \( v(k) \) in (6). Similarly, the output \( y(k) \) can be derived from (1), (3) and (6) as [15]
\[ y(k) = AB \delta_0(k) \cos(\omega_o k + \theta - \phi) - \rho AB^2 \delta_0^2(k) \cos(\omega_o k + \theta - 2\phi) + v_1(k) \] (15)

where \( v_1(k) \) is a noise term in \( y(k) \) due to input noise \( v(k) \) in (6). By taking the expectation of (8), we obtain the difference
\[ v = E[\delta_0(k + 1)] = E[\delta_0(k)] - \mu E[y(k)x_2(k)] \] (16)

Substituting (14) and (15) into (16) yields
\[ E[\delta_0(k + 1)] \approx \left( 1 - \frac{1}{2} \mu A^2 B^2 \right) E[\delta_0(k)] - \mu E[v_1(k)v_2(k)] \] (17)

where the terms including \( \delta_i^2(k) \) for \( i \geq 2 \) are omitted. The stability condition of the system in (17) is found to be
\[ \left| 1 - \frac{1}{2} \mu A^2 B^2 \right| < 1, \quad \text{i.e.} \ 0 < \mu < \frac{4}{A^2 B^2}. \] (18)

When \( v(k) \) is a white noise, \( E[v_1(k)v_2(k)] \) can be obtained from the complex integration along unit circle in the \( z \)-plane as
\[ E[v_1(k)v_2(k)] = \frac{\sigma_v^2}{2\pi i} \oint \frac{H(z)F_2(z^{-1})}{z} \frac{dz}{z} = \frac{\sigma_v^2}{2\pi i} \oint \frac{z^2 + a_0 z + 1}{(z - z_1)(z - z_2)} \frac{dz}{z} = \sigma_v^2 \text{Res}(z_1) + \text{Res}(z_2) \] (19)

where \( z_1 \) and \( z_2 \) denote the roots of \( z^2 + a_0 z + \rho^2 = 0 \) and the residues are written as
\[ \text{Res}(z_1) = \frac{1}{1 + \rho} \cdot \frac{a_0 z_1 + 1 + \rho}{(z_1 - z_2)(1 - z_1^2)} \]
\[ \text{Res}(z_2) = \frac{1}{1 + \rho} \cdot \frac{a_0 z_2 + 1 + \rho}{(z_1 - z_2)(1 - z_2^2)} \]
due to \( z_i^2 + a_0 z_i + 1 = (1 - \rho)(a_0 z_i + 1 + \rho) \) for \( i = 1, 2 \). Noting that \( z_1 + z_2 = -\rho a_0 \) and \( z_1 z_2 = \rho^2 \), we obtain
\[ E[v_1(k)v_2(k)] = \frac{1 - \rho}{1 + \rho} \cdot \frac{a_0}{\rho^4 - 2\rho^2 \cos 2\omega_o + 1} \sigma_v^2 \]
\[ = \frac{1 - \rho}{1 + \rho} \cdot \frac{2 \cos \omega_o}{\rho^4 - 2\rho^2 \cos 2\omega_o + 1} \sigma_v^2. \] (20)

If the first-order linear system in (17) is stable, as \( k \to \infty \)
\[ E[\delta_0(\infty)] = -\frac{\mu}{\frac{7}{2} \mu A^2 B^2} E[v_1(k)v_2(k)] \]
\[ = \frac{1}{A^2 B^2} \cdot \frac{1 - \rho}{1 + \rho} \cdot \frac{4 \cos \omega_o}{\rho^4 - 2\rho^2 \cos 2\omega_o + 1} \sigma_v^2. \] (21)

This is an analytical closed-form expression for the estimation bias of parameter \( a(k) \) in the adaptive state-space notch digital filter in (7) and (8). The parameter-estimation bias in (21) is much simpler than that given by (19) and (29) in [15] which is derived from (11) and a function of parameter \( \mu \).

IV. A Numerical Example

The amplitude and phase of the sinusoid are set to \( A = 1 \) and \( \theta = \pi/4 \) in (6), respectively. The variance of the white noise \( v(k) \) is set to \( \sigma_v^2 = 0.05 \) in (6). The frequency of the sinusoid and the initial estimation-parameter are set to \( \omega_o = 0.3\pi \) and \( a(0) = -2\cos(0.2\pi) \), respectively. In addition, \( x(0) = 0 \) and \( \rho = 0.96 \) are chosen in (7a) and (7b), respectively.

When the simplified iterative algorithm (SIA) in (8) is applied to this example, the convergence characteristics of the parameter \( a(k) \) are shown in Fig. 2 for \( \mu = 0.0015 \), \( 0.0010 \), \( 0.0005 \). From Fig. 2, it is observed that \( a(k) \) at the steady state is close to the true value \( a_0 = -2\cos(0.3\pi) \) for all the values of fixed \( \mu \).

Moreover, the MSEs of the convergence characteristics of the parameter \( a(k) \) for the SIA in (8) with \( \mu = 0.0015 \) fixed and a VSS algorithm with \( \mu = 0.0015 \), \( \xi = 0.98 \) and \( \eta = 0.0002 \) in (9) are plotted in Fig. 3. The plots are obtained by ensemble average of 100 independent runs, and the MSEs at time \( k \) are computed using
\[ MSE = \frac{1}{100} \sum_{i=1}^{100} (a(k)_i - a_0)^2 \]

where \( a(k)_i \) indicates the parameter \( a(k) \) derived from \( i \)th run at time \( k \). Fig. 3 shows that the proposed state-space notch filter converges to \( a_0 = -2\cos(0.3\pi) \) with high estimation accuracy for the fixed value \( \mu \), especially for the VSS \( \mu(k) \).

Regarding the parameter-estimation bias at steady state, experimental results are compared with the theory based on (21) in Fig. 4 where data \( E[a(k)] \) are the ensemble average.
of 100 independent runs at $k = 2000$ obtained using the VSS algorithm with $\mu(0) = 0.0015$, $\xi = 0.98$ and $\eta$ being in the range $[0.0001, 0.0007]$ in (9).

From Fig. 4, it is observed that the experimental results are close to the theoretical ones for all the discrete values of $\rho$.

V. CONCLUSION

This paper has explored a state-space approach for adaptive second-order IIR notch digital filters with constrained poles and zeros. A simplified adaptive iterative algorithm has been derived from the gradient-descent method to minimize the mean-squared output error of a second-order state-space notch digital filter. Moreover, the stability and parameter-estimation bias of the adaptive iterative algorithm have been analyzed, and an analytical closed-form expression for the estimation bias of parameter $a(k)$ has been derived. The results of a numerical example have demonstrated the validity and effectiveness of the proposed adaptive state-space notch digital filter and the parameter-estimation bias analysis. It is noted that in [15] two difference equations, i.e., a second-order linear dynamical system, on the mean and mean-squared errors were established to analyze stability and parameter-estimation bias. Hence, the stability and parameter-estimation bias analysis in this paper which relies on a first-order linear dynamical system is much simpler than that in [15] dependent on step-size parameter $\mu$.

REFERENCES